Short Papers

Scattering Matrix Description of Microwave Resonators

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Abstract —**In this paper an analysis of the scattering matrix coefficients (reflectances and transmittances) of the microwave resonators is presented. Two general cases of resonator couplings are considered. One of them is a multiport with one port terminated by the resonator, the other is a resonator coupled, in various ways, to a number of transmission lines. The effect of external circuits on the Q and resonant frequency of the resonator has also been examined. Fundamental parameters for the commonly used microwave resonator couplings are shown in Table I.**

I. Introduction

The operation and properties of microwave resonators may be presented in various ways. The form of the description is chosen according to needs, and it also depends on the possibilities offered by the given form. The following three basic forms may be distinguished.

Field description is the most universal and rigorous one and is based on the Maxwell equations. It is possible to solve these equations for many various configurations of the boundary surfaces of the resonator and various filling materials and, thus, to determine such parameters as the Q's or resonant frequencies [2].

In the *circuit description*, equivalent circuits with lumped constants typical of the circuit theory are used. The values of the equivalent circuit elements are related in a simple way to the field defined parameters such as resonant frequencies and Q's [1], [2], [4], [6].

Scattering-matrix description is relatively little known and not very popular yet. It consists in determining the coefficients (reflectances and transmittances) of the scattering-matrix of a microwave circuit including a resonator. These coefficients can be expressed as the functions of frequency, the quality factor Q, and the coupling parameter of the resonator and the exciting waveguide. These functions can be determined on the basis of the field [5], [7], or circuit [3] descriptions. It can be shown that the use of matrix description simplifies the analysis of the resonator parameters measurements methods [3], and the analysis of the oscillation conditions for some types of generators. It also helps one to foresee the operation of complex circuits with a resonator contained in them. The aim of this paper is to show that the reflectances and transmittances of a common and relatively complex microwave circuit with a resonator are simple functions of frequency having forms identical with those for the simplest case of the single-ended resonator.

These functions will be derived—assuming that:

a) the loaded Q of the resonator is high;
b) the functions are valid within the narrow frequency range near the resonant frequency, the resonant frequencies of the other modes are far outside this range; and
c) in this narrow frequency range the nonresonant parameters of the external circuits are constant.

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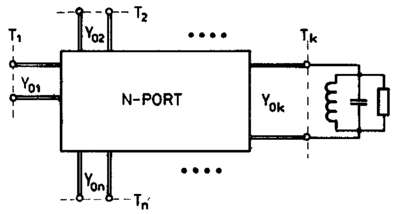

Fig. 1. The single-ended resonator connected to the kth port of an n-port.

II. Multiport Structure with One Port Loaded by Resonator

The connection of a single-ended resonator to the linear multiport is an important case in microwave practice (Fig. 1). The resonator, connected to the kth port, is represented by a simple low-frequency parallel resonance curcuit. The T_k plane is the so-called detuned short plane. The normalized (in relation to the characteristic admittance Y_{0k} of a guide) admittance y_r of the resonator is expressed by the following formula:

$$y_r = \frac{1}{\beta} + j\alpha\left(\frac{1+\beta}{\beta}\right) \tag{1}$$

where β is the coupling parameter [4], [7], α is named by Altman [1] a normalized frequency, and by Sucher [7] a bandwidth parameter, depends on the loaded Q_L, resonant frequency f_0 and frequency f

$$\alpha = Q_L\left(\frac{f}{f_0} - \frac{f_0}{f}\right) \cong 2Q_L\frac{f-f_0}{f_0}. \tag{2}$$

The reflection coefficient $\Gamma(\alpha, T_k)$ of the resonator, determined in the plane T_k is given as

$$\Gamma(\alpha, T_k) = \frac{1-y_r}{1+y_r} = -1 + \frac{D}{1+j\alpha} \tag{3}$$

where, in this case, D is a real and positive number, dependent on β or the ratio of the Q_L and external Q, Q_e

$$D = \frac{2\beta}{1+\beta} = \frac{2Q_L}{Q_e}. \tag{4}$$

In the complex plane Γ is a well-known circle, suspended at point -1, and D is its favored diameter.

On the basis of the n-port scattering matrix coefficients and the function $\Gamma(\alpha)$ of the resonator, the scattering matrix coefficients of the resulting $(n-1)$-port can thus be determined as

$$S'_{ji} = S_{ji} + \frac{S_{ki}S_{jk}\Gamma}{1-S_{kk}\Gamma}. \tag{5}$$

In the above formula, S_{ji}, S_{ki}, S_{jk}, and S_{kk} are the coefficients of the n-port matrix, and S'_{ji} is the coefficient of the $(n-1)$-port. The above formula (5) can be transformed into a more general one

$$S'_{ji} = S_0 + \frac{D'}{1+j\alpha'}. \tag{6}$$

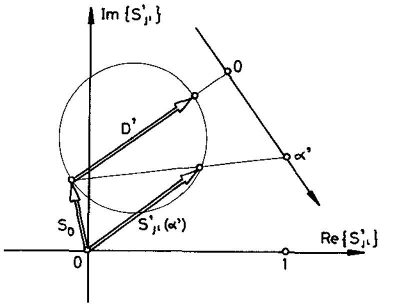

Fig. 2. The circle of the n-port scattering matrix coefficient the one port of which is terminated by the resonator.

The values of S_0 and D' can easily be found

$$S_0 = S_{ji} - \frac{S_{ki}S_{jk}}{1+S_{kk}} \tag{7}$$

$$D' = \frac{DS_{ki}S_{jk}e^{-j2\Omega}}{|1+S_{kk}|^2 - D\left(|S_{kk}|^2 + \mathrm{Re}\{S_{kk}\}\right)} \tag{8}$$

where $\Omega = \mathrm{Arg}\{1+S_{kk}\}$. The new resonant frequency is determined by

$$\alpha - \frac{\mathrm{Im}\{DS_{kk}\}}{|1+S_{kk}|^2} = 0. \tag{9}$$

A change of the resonant frequency can be observed only when $\mathrm{Im}\{S_{kk}\} \neq 0$. This means that if the resonator is excited by a source not matched to the guide connecting them, the energy stored in the resonator will be at its maximum for a frequency slightly different from that resulting merely from its dimensions. If $S_{kk} \neq 0$, a change of Q_L should also be expected. The resulting Q'_L is then equal to

$$Q'_L = \frac{Q_L|1+S_{kk}|^2}{|1+S_{kk}|^2 - D\left(|S_{kk}|^2 + \mathrm{Re}\{S_{kk}\}\right)}. \tag{10}$$

The position of the circle $S'_{ji}(\alpha')$ in the complex plane is shown in Fig. 2. The circle is shifted by S_0 in relation to the coordinate origin and suspended at the end of phasor S_0. D' is the favored diameter of the circle, axis α' is perpendicular to D' and distant by 1 from circle suspension point, the scale of axis α' is the same as the scales of the real and imaginary axes. The simple graphical construction leads to the determination of phasor $S'_{ji}(\alpha')$.

III. General Case of Resonator Coupling

In practice, the resonator can be coupled to a number of lines in various ways (Fig. 3). The scattering matrix coefficients of such a multiport can be determined in terms of the proper equivalent circuit which will contain a parallel resonant circuit. The normalized admittance y_r of that circuit can be expressed by (1). The coefficient S_{nm} is the bilinear function of y_r, and, consequently of α. Thus, in general, we can write

$$S_{nm} = \frac{W_1 + j\alpha W_2}{W_3 + j\alpha}. \tag{11}$$

Where W_1, W_2, and W_3 are complex coefficients independent of the frequency. It can be seen that (11) may be expressed in the same form as (3) and (6)

$$S_{nm} = S_0 + \frac{D}{1+j\alpha'}. \tag{12}$$

In general, S_0 and D are complex numbers

$$S_0 = W_2 \tag{13}$$

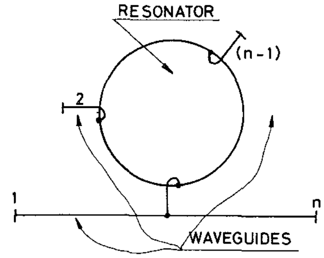

Fig. 3. The resonator coupled to the waveguides in various ways.

and

$$D = \frac{W_1 - W_2W_3}{\mathrm{Re}\{W_3\}}. \tag{14}$$

The normalized frequency α' differs from α

$$\alpha' = \frac{\alpha + \mathrm{Im}\{W_3\}}{\mathrm{Re}\{W_3\}}. \tag{15}$$

We can see from the last equation that connections between the lines and the resonator cause a change of the resonant frequency and the Q_L, as well. It can also be noted that for the new resonance frequency, for which $\alpha' = 0$, condition (16) must be fulfilled

$$\left|\frac{dS_{nm}}{df}\right| = \text{maximum}. \tag{16}$$

The last general equation makes it easier to interpret the observed characteristics of the real circuits.

IV. Simple Cases of Resonator Couplings

In microwave practice, four commonly used cases of resonator couplings with the transmission lines have been known. The single-ended resonator is a one-port structure, and its reflection coefficient $\Gamma(\alpha)$ is determined by (3) and (4).

The transmission resonator (TR) can be characterized by the scattering matrix

$$[S] = \begin{bmatrix} R_1 & T \\ T & R_2 \end{bmatrix}. \tag{17}$$

The values S_0 and D for the coefficients T, R_1, and R_2 can be taken from Table I in the case when the detuned short planes are chosen as the reference planes [3], [5]. β_1 and β_2 are the coupling parameters.

The reaction-type resonator (RR) is a symmetrical two-port structure and its scattering matrix is given by

$$[S] = \begin{bmatrix} R & T \\ T & R \end{bmatrix}. \tag{18}$$

The values S_0 and D for T and R can be found in Table I in the case when the equivalent circuit of the resonator is represented by a parallel resonance circuit coupled in series with the line.

The reaction-transmission type resonator (RTR) is a three-port unit obtained when its coupling with the main line is of the reaction type, and is additionally coupled with an auxiliary line (Fig. 4). The scattering matrix of the three-port from Fig. 4 can be described as follows:

$$[S] = \begin{bmatrix} R_M & T_M & T_{MA} \\ T_M & R_M & -T_{MA} \\ T_{MA} & -T_{MA} & R_A \end{bmatrix}. \tag{19}$$

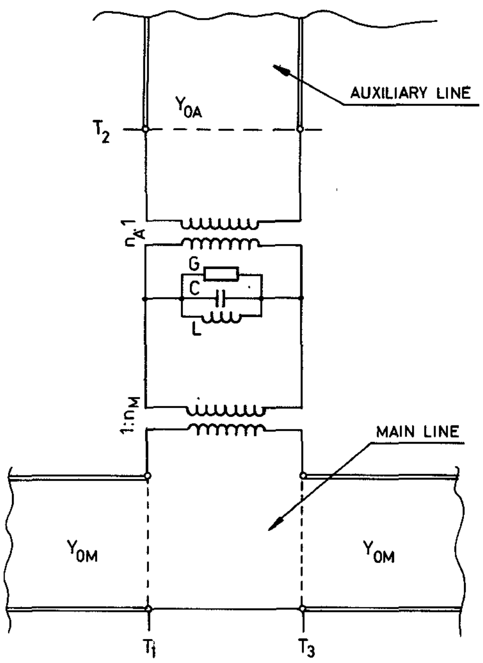

Fig. 4. The equivalent circuit of the reaction-transmission type resonator.

TABLE I
S_0 AND D VALUES FOR SCATTERING MATRIX COEFFICIENTS OF SIMPLE RESONATOR COUPLINGS

TYPE OF COUPLING	COEF-FICIENT	S_0	D
TR	T	0	$2\sqrt{\beta_1\beta_2}(1+\beta_1+\beta_2)^{-1}$
	R_1	-1	$2\beta_1(1+\beta_1+\beta_2)^{-1}$
	R_2	-1	$2\beta_2(1+\beta_1+\beta_2)^{-1}$
RR	T	1	$-\beta(1+\beta)^{-1}$
	R	0	$\beta(1+\beta)^{-1}$
RTR	T_M	1	$-\beta_M(1+\beta_A+\beta_M)^{-1}$
	T_{MA}	0	$\sqrt{\beta_A\beta_M}(1+\beta_A+\beta_M)^{-1}$
	R_M	0	$\beta_M(1+\beta_A+\beta_M)^{-1}$
	R_A	-1	$2\beta_A(1+\beta_A+\beta_M)^{-1}$

The values S_0 and D for T_M, T_{MA}, R_M, and R_A are shown in the Table I. β_M and β_A are the coupling parameters with the main line and auxiliary one.

The last type of resonator is often used in the oscillators.

V. Conclusions

As it has been shown in this paper, the reflectances and transmittances of the microwave multiports with resonators are simple and identical as regards the form of their dependence on the frequency. In the complex plane, the graphs of these functions have the form of circles, their positions are determined by phasors S_0 (suspension point) and D (favored diameter).

The above presented matrix form of the resonator properties description is expected to simplify and facilitate the operation conditions of devices cooperating with these resonators.

References

[1] J. L. Altman, *Microwave Circuits*. Princeton: Van Nostrand, 1964, ch. 5.
[2] R. E. Collin, *Foundations for Microwave Engineering*. New York: McGraw-Hill, 1966, ch. 7.
[3] B. A. Galwas, *Homodyne Methods of Measuring Microwave Circuits*. Warsaw, Poland: Warsaw Techn. Univ. Press (in Polish), 1976, ch. 5.
[4] E. J. Ginzton, *Microwave Measurements*. New York: McGraw-Hill, 1957, ch. 9.
[5] A. E. Pannenborg, "On the scattering matrix of symmetrical waveguide junctions," *Philips Res. Rep.*, vol. 7, no. 2, pp. 131–157, Apr. 1952.
[6] M. Sucher and J. Fox, *Handbook of Microwave Measurements*. New York: Polytechnic Press, 1963, vol. II, ch. VIII.
[7] S. Tomonaga, "A general theory of ultra-short-wave circuits II," *J. Phys. Soc. Jap.*, vol. 3, pp. 93–105, Mar.–June 1948.

Performance of Microstrip Couplers on an Anisotropic Substrate with an Isotropic Superstrate

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***Abstract*—The directivity of microstrip couplers on anisotropic substrates can be improved substantially by adding an isotropic superstrate layer above the microstrips. The necessary dimensions of the layer can be chosen so that fabrication is simple and noncritical.**

I. Introduction

Directivity of microstrip directional couplers is strongly dependent upon the equality of even- and odd-mode phase velocities. These quantities are unequal for simple microstrip, and the inequality becomes more severe for substrates of high dielectric constant. For coupling values less than 6 dB, and dielectric constants below 10, unequal phase velocities rarely limit directivity. However, for weaker coupling, and especially on certain anisotropic substrates, directivity degradation can be severe. This is particularly true of sapphire, wherein the dielectric constant parallel to the optical axis is greater than that in the perpendicular direction.

Directivity can be improved substantially by adding another high-dielectric layer on top of the microstrip substrate. This "superstrate" effectively slows the odd mode relative to the even mode. In some cases, it is possible to equalize phase velocities, and it is almost always possible to improve directivity.

In previous efforts involving isotropic layers, Sheleg and Spielman [1] presented an iterative design technique. Paolino [2] and Haupt and Delfs [3] presented an analytical formulation and experimental data for similar couplers on isotropic substrates. Karekar and Pande [4] report directivity improvement using a

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